Grey-box modelling of a friction affected dynamical system

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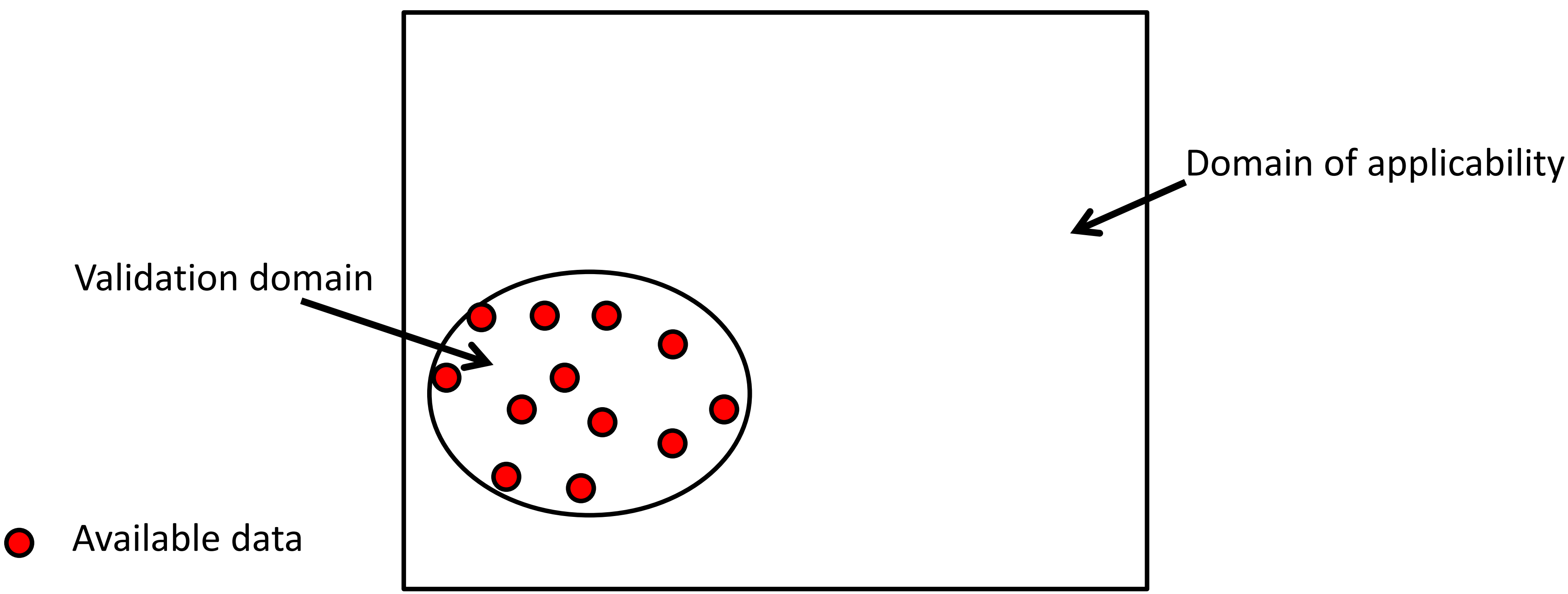
Introduction

Decision making is increasingly reliant on computational models. These models are often calibrated and validated using sets of training data.

The problem: often, we then use these models to make predictions in regions where testing is difficult / expensive.

We extrapolate outside of the **validation domain**:



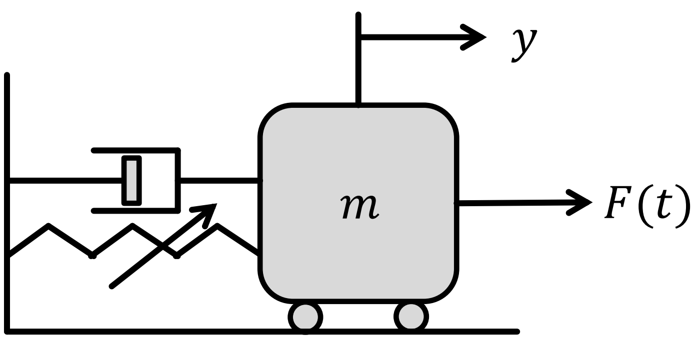
Examples:

- Different regions of the input domain: predicting seismic response using a model validated on ambient vibration data.

- Different response features: a model validated using modal data is used to predict maximum stresses.

How much confidence can we place in these predictions ?

Consider this equivalent linearisation problem, where F is white noise with PSD height S.



True system: $m\ddot{y} + c\dot{y} + ky + k_3 y^3 = F(t)$

Model: $m\ddot{y} + c\dot{y} + k_{eq} y = F(t)$

$$k_{eq} = f(S)$$

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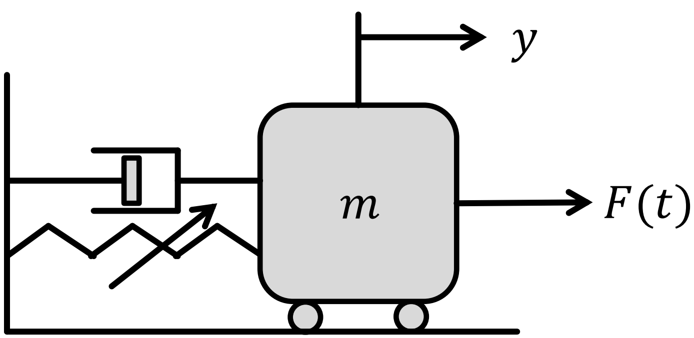

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Changes with excitation power – cannot extrapolate to higher excitation levels.

Perhaps quantifying parameter uncertainty is relatively unimportant. The problem here is **model error**.

From [1]:

$$z_i = y_i(\boldsymbol{x_i}) + \eta_i(\boldsymbol{x_i}) + \epsilon_i$$

z_i - measurement
y_i - output of physical law based model
η_i - **model error**
ϵ_i - measurement noise

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Grey box model – combination of white and black box.

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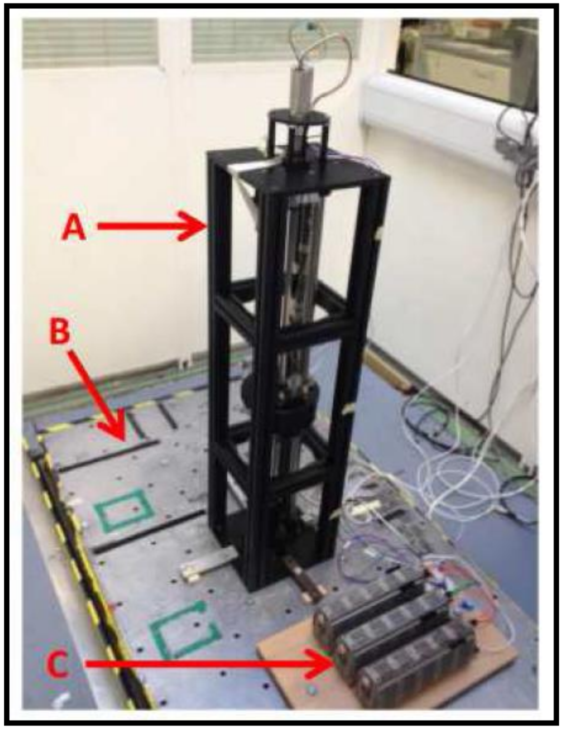
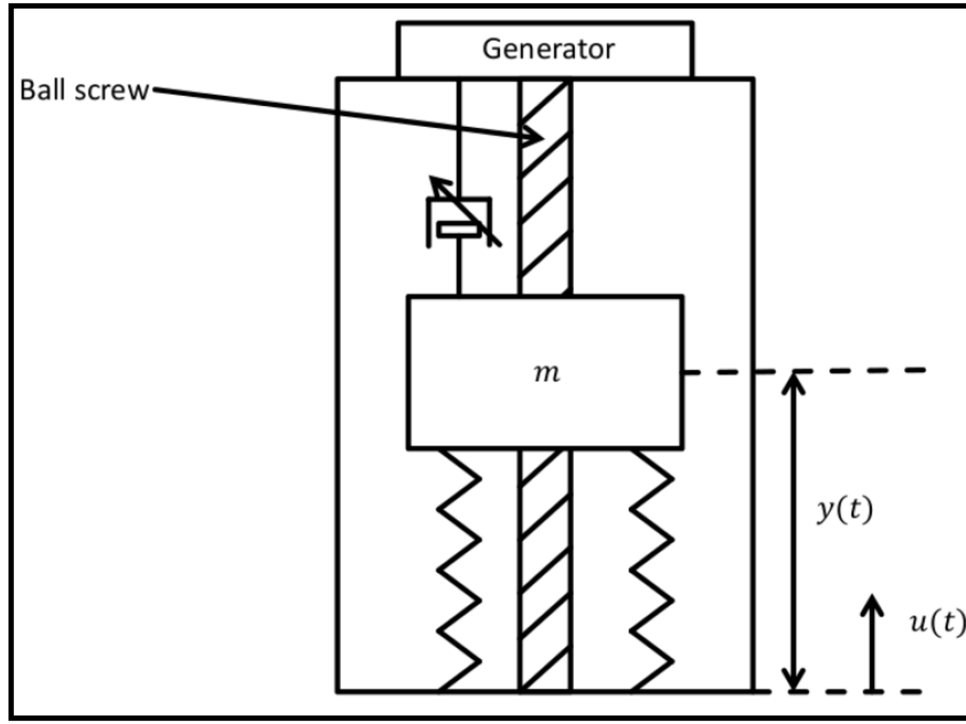
Case study





White box model:

$$M\ddot{y} + b_m\dot{y} + ky + F_c \tanh(\beta\dot{y}) = -m\ddot{u}$$

$$M = m + J\left(\frac{2\pi}{l}\right)^2 \qquad b_m = \left(\frac{2\pi}{l}\right)^2 c_m$$

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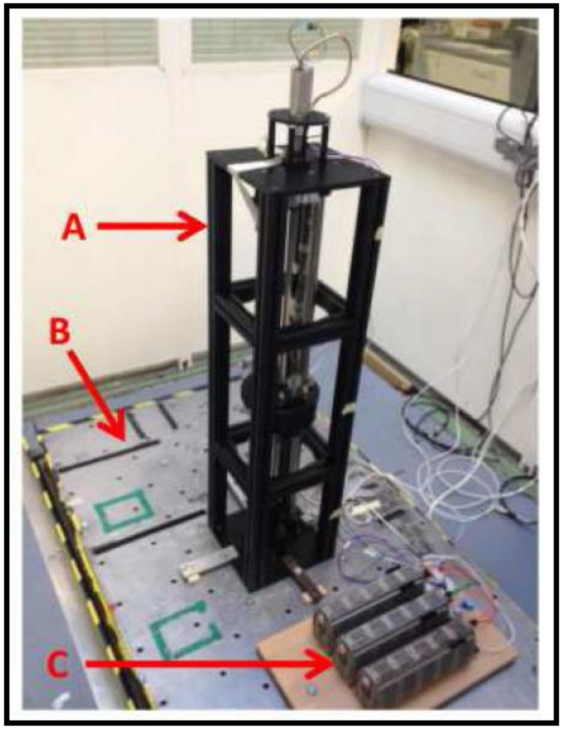

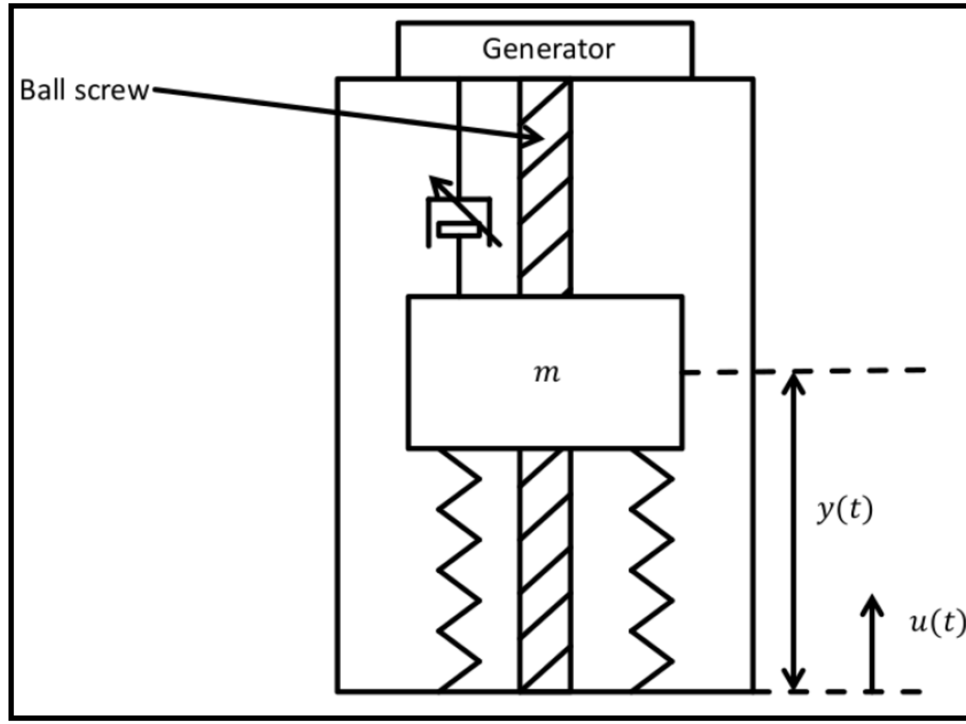

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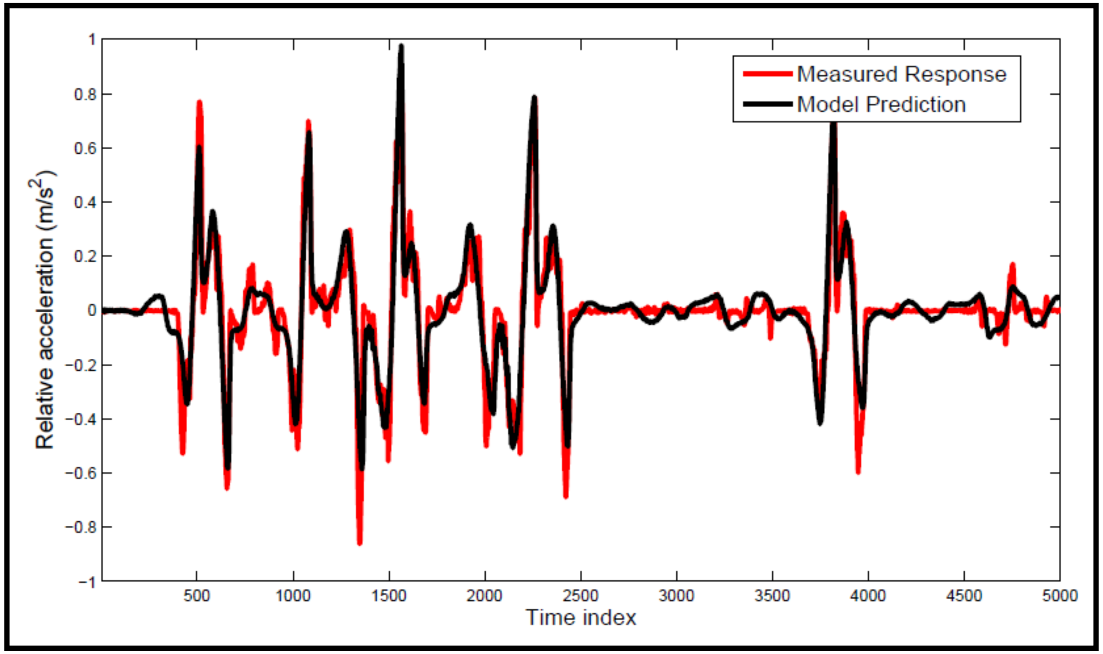
Bayesian system identification used to infer probabilistic estimates from acceleration time histories [2].

Predictions appeared to be robust against parameter uncertainties:



But what about **model error**?

[2] Green, P. L., Hendijanizadeh, M., Simeone, L., & Elliott, S. J. (2015). Probabilistic modelling of a rotational energy harvester. *Journal of Intelligent Material Systems and Structures*, 1045389X15573343.

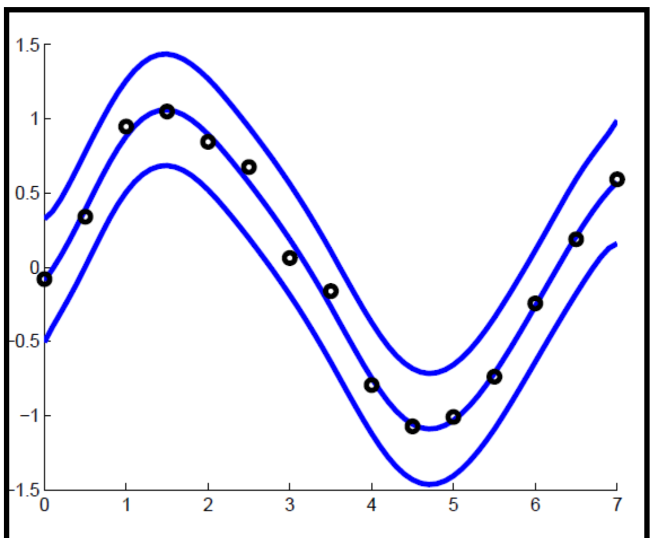
Here it is defined as

$$\eta_i(\boldsymbol{x_i}) = \ddot{z}_i - \ddot{y}_i, \quad i = 1, \dots, N$$

$\ddot{z}_i$ is the measured relative acceleration, $\ddot{y}_i$ is the relative acceleration according to the white box model and $\boldsymbol{x_i}$ is an input (to be defined).

We will train a machine learning algorithm such that, if it sees a new input $\boldsymbol{x}^*$, it will product a new estimate of model error, η^*.

In this case we used a Gaussian process (GP). Skipping some details, we can simply think of these an interpolators which, conveniently, also output confidence bounds:



But what is the input, x?

Previous work has modelled static relationships between system inputs and model error. The system described here is **dynamical**. It is therefore **autoregressive**.

Here then, we hypothesise that model error must be emulated using a **NARX model:**

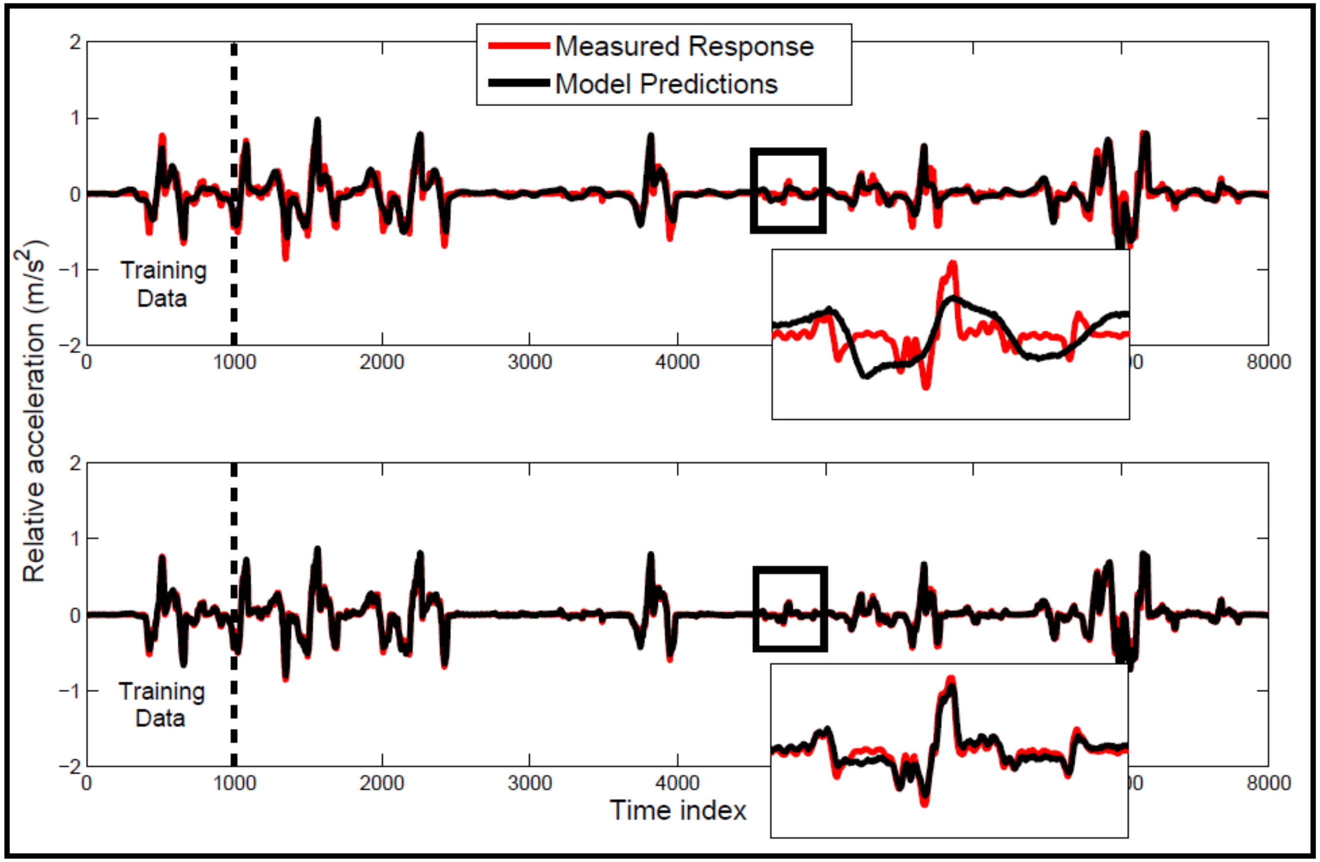
$$\eta_i^* = f(\eta_{i-1}, \eta_{i-2}, \dots, u_i, u_{i-1}, u_{i-2})$$

Previous model error and previous excitation (we can include current excitation too).

GP NARX model predicts η_i^* using the input:

$$\boldsymbol{x_i} = \begin{pmatrix} \eta_{i-1} \\ \eta_{i-2} \\ \vdots \\ u_i \\ u_{i-1} \\ \vdots \end{pmatrix}$$

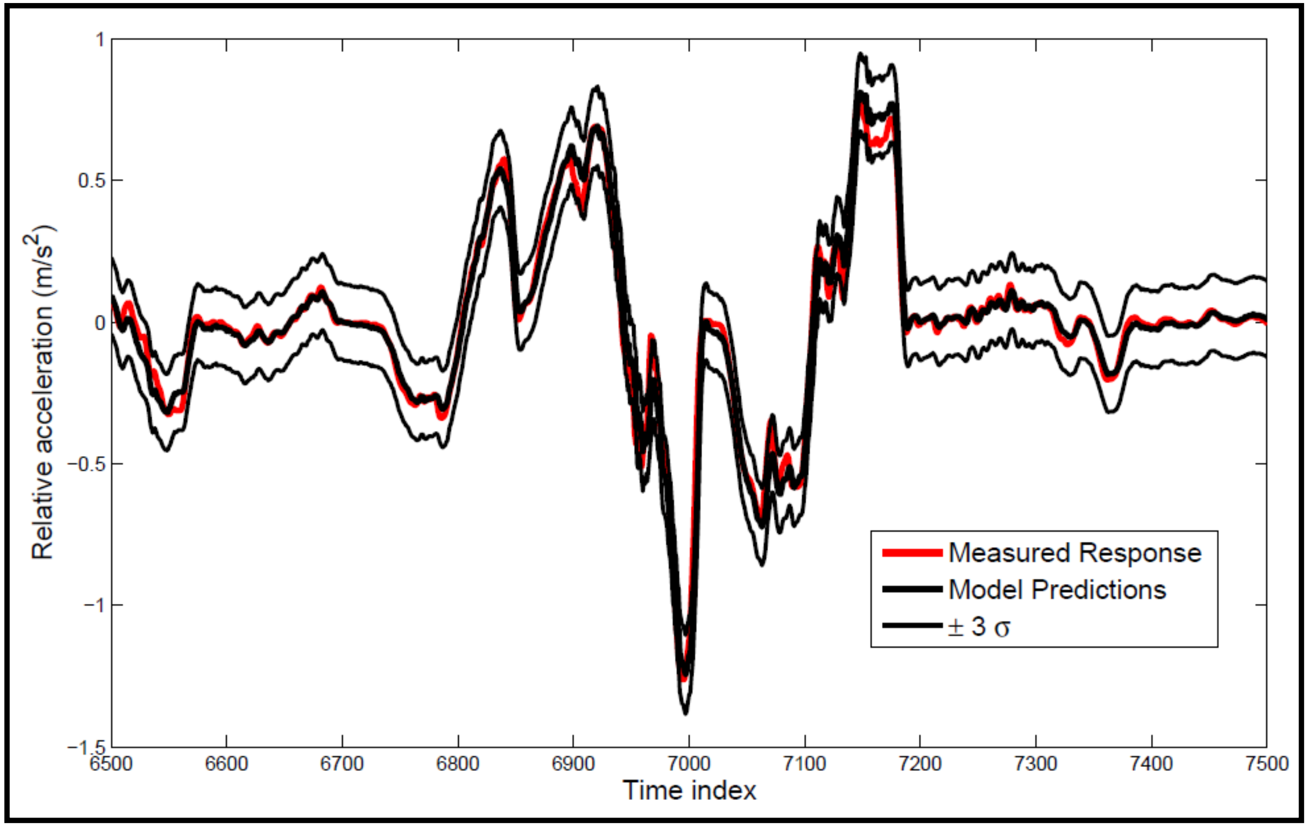
For now we will only consider one at a time predictions…



GPs also come equipped with a kind of **Automatic Relevance Determination** to help identify key input parameters. These predictions were actually made using inputs:

$$\boldsymbol{x_i} = \begin{pmatrix} \eta_{i-1} \\ \eta_{i-2} \\ u_i \end{pmatrix}$$

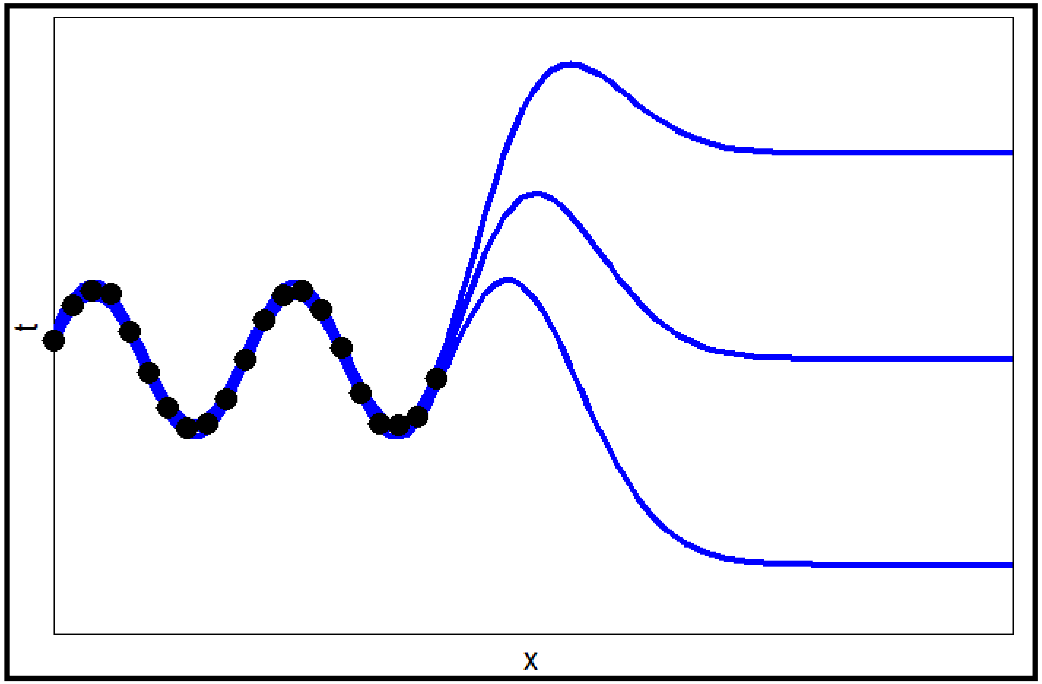
We also get confidence bounds – quantifying the uncertainties associated with **model error:**



Discussion

What about extrapolation (larger amplitude excitation, for example)?

If you extrapolate far from your training data, you become increasingly reliant on the **prior** specification of the GP. Example where the prior was zero mean:

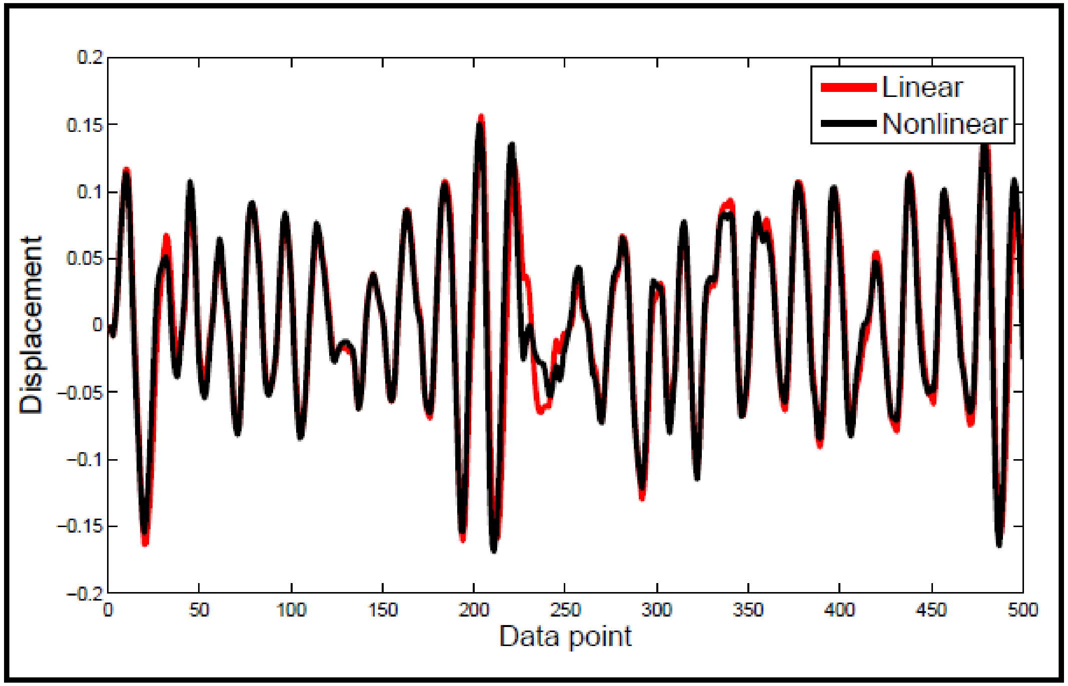


There is no convenient solution here – the confidence bounds don't necessarily reflect how far you are from the validation domain.

But how far are we really extrapolating?

Our input space is now multidimensional. It is a complex mix of previous measurements of model error and excitation – we may not be extrapolating as far as we think.

Some preliminary results will be presented in [3]. Here we deliberately tried to model a Duffing oscillator using a linear system. **Low amplitude training data:**



[3] Green, P. L. The diagnosis and simulation of discrepancies in dynamical models. *Proceedings of IMAC XXXIV, a conference and exposition on structural dynamics*. (Jan 2016)

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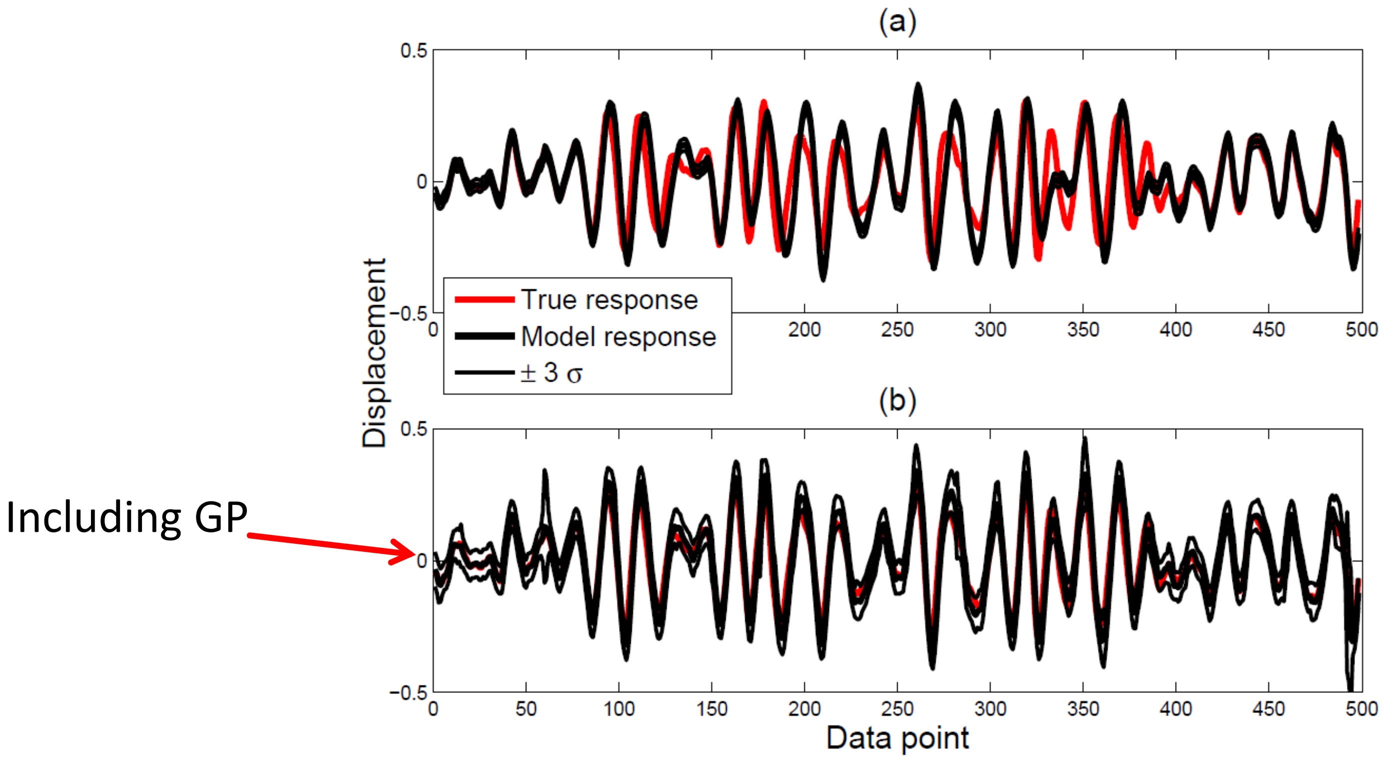

[3] Green, P. L. The diagnosis and simulation of discrepancies in dynamical models. *Proceedings of IMAC XXXIV, a conference and exposition on structural dynamics*. (Jan 2016)

We only studied one at a time predictions here.

With ‘full model predictions’, the inputs to the GP would be its own, uncertain, previous predictions:

$$\boldsymbol{x_i} = \begin{pmatrix} \eta^*_{i-1} \\ \eta^*_{i-2} \\ \vdots \\ u_i \\ u_{i-1} \\ \vdots \end{pmatrix}$$

This will cause a large source of uncertainty which will grow as we move further away from the most recent measure of model error.

An interesting question: how can we mitigate this source of uncertainty?

Alternatively: can we use the GP to help learn about the physics of real systems?

In this simple case, we can (and did) plot the **restoring force surface:**

$$M\ddot{y} + b_m\dot{y} + ky + F_c\tanh(\beta\dot{y}) + f(y,\dot{y}) = -m\ddot{u}$$

$$\therefore f(y,\dot{y}) = -(m\ddot{u} + M\ddot{y} + b_m\dot{y} + ky + F_c\tanh(\beta\dot{y}))$$

but this is difficult for large, MDOF systems.

We can always try least-squares approaches but, as part of this, we have to propose a **parametric** emulator of model error. Example:

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Gaussian process are **nonparametric**.

They allow us to identify which inputs are important (ARD). This may help to identify hysteresis or important correlations between degrees of freedom.

They can also **emulate large numbers of outputs** (with the help of a Singular Value Decomposition) [4].

[4] Higdon, D., Gattiker, J., Williams, B., & Rightley, M. (2008). Computer model calibration using high-dimensional output. *Journal of the American Statistical Association*, *103*(482).

Thank you for listening

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